

Topology and Combinatorics of Partitions of Masses by Hyperplanes

Peter Mani–Levitska*
Mathematics Institute, Bern

Siniša Vrećica†
Faculty of Mathematics, Belgrade

Rade Živaljević‡
Mathematics Institute SANU, Belgrade

October 2003

Abstract

An old problem in combinatorial geometry is to determine when one or more measurable sets in \mathbb{R}^d admit an *equipartition* by a collection of k hyperplanes, [19] [20]. The problem can be reduced to the question of (non)existence of a map $f : (S^d)^k \rightarrow S(U)$, equivariant with respect to the Weyl group $W_k := (\mathbb{Z}/2)^{\oplus k} \rtimes S_k$, where U is a representation of W_k and $S(U) \subset U$ the corresponding unit sphere. In this paper we develop a general method for computing topological obstructions for the existence of such equivariant maps. Emphasizing the combinatorial point of view, we show that the computation of relevant cohomology/bordism obstruction classes can be in many cases reduced to the question of enumerating the classes of immersed curves in \mathbb{R}^2 with a prescribed type and number of intersections with the coordinate axes. It turns out that the last problem is closely related to some “cyclic word enumeration problems” from enumerative combinatorics which are usually approached via Polyá enumeration or Möbius inversion technique. Among the new results is the well known open case of 5 measures and 2 hyperplanes in \mathbb{R}^8 , [33]. The obstruction in this case is identified as the element $2X_{ab} \in H_1(\mathbb{D}_8; \mathbb{Z}) \cong \mathbb{Z}/4$, where X_{ab} is a generator, which explains why this result cannot be obtained by the parity count formulas from [33] or the methods based on either Stiefel-Whitney classes or ideal valued cohomological index theory [17].

*Supported by Schweizerischer Nationalfonds zur Förderung der wissenschaftlichen Forschung.

†Supported by the Ministry for Science and Technology of Serbia, Grant 1643.

‡Supported by the Ministry for Science and Technology of Serbia, Grant 1643.

Contents

1	Introduction	3
1.1	The Equipartition Problem	3
1.2	History of the Problem	4
1.3	Mass Distributions	4
2	Proof Technique	5
2.1	The General Scheme of the Proof	5
2.2	An Approach Based on \mathbf{G} -bordism	5
2.3	Equivariant Maps	6
2.3.1	The central paradigm	6
2.3.2	Equipartition problem revisited	7
2.3.3	Equivariant Poincaré duality	8
2.4	Moment Curve	9
2.5	Circular $\{\mathbf{A}, \mathbf{B}\}$ -words	9
2.6	Homology and Bordism Computations	10
2.6.1	Dihedral group \mathbb{D}_8	11
2.6.2	Homology computations	11
2.6.3	Geometric interpretation	13
2.6.4	The algorithm	14
3	Results And Proofs	15
3.1	The case $\Delta = 0$	15
3.2	The case $\Delta = 1$	16
3.2.1	Types and inventory of solutions	16
3.2.2	The $(8, 5, 2)$ case	20
3.2.3	The case $(6m + 2, 4m + 1, 2)$	21
3.2.4	The case $(6m - 1, 4m - 1, 2)$	23
4	Cohomological Methods	23
4.1	Ideal valued cohomological index theory	23

1 Introduction

1.1 The Equipartition Problem

Definition 1.1 Suppose that

$$\mathcal{M} = \{\mu_1, \mu_2, \dots, \mu_j\}$$

is a collection of continuous mass distributions/measures defined in \mathbb{R}^d . If $\mathcal{H} = \{H_i\}_{i=1}^k$ is a collection of k hyperplanes in \mathbb{R}^d in general position, the connected components of the complement $\mathbb{R}^d \setminus \bigcup \mathcal{H}$ are called (open) k -orthants. The definition can be meaningfully extended to the case of degenerated collections \mathcal{H} , when some of the k -orthants are allowed to be empty.

A collection \mathcal{H} is an equipartition, or more precisely a k -equipartition for \mathcal{M} if

$$\mu_i(O) = \mu_i(\overline{O}) = \frac{1}{2^k} \mu_i(\mathbb{R}^d)$$

for each of the measures $\mu_i \in \mathcal{M}$ and for each k -orthant O associated to \mathcal{H} .

Definition 1.2 A triple (d, j, k) of integers is referred to as admissible if for any collection $\mathcal{M} = \{\mu_i\}_{i=1}^j$ of j continuous measures in \mathbb{R}^d , there exists a collection of k hyperplanes $\mathcal{H} = \{H_i\}_{i=1}^k$ forming an equipartition for all measures in \mathcal{M} .

Problem 1.3 The general problem is to characterize the set \mathcal{A} of all admissible triples. If the emphasis is put on the ambient Euclidean space \mathbb{R}^d , the equivalent problem is to determine the smallest dimension $d := \Delta(j, k)$ such that the triple (d, j, k) is admissible.

One of the main results of the paper is the following theorem which gives a sufficient condition for a triple $(6m+2, 4m+1, 2)$ to be admissible.

Theorem 1.4 Suppose that $(d, j, k) = (6m+2, 4m+1, 2)$ where m is a positive integer. Then there exists an equipartition of $j = 4m+1$ mass distributions in $\mathbb{R}^d = \mathbb{R}^{6m+2}$ if

$$\Omega(m) := \alpha(2m+1) + 2\beta(2m+1) - \gamma(2m+1)$$

is not divisible by 4, where α, β, γ are the combinatorial functions counting special classes of cyclic, signed $\{A, B\}$ -words, introduced in Definition 3.10.

It turns out (Theorem 3.9) that $\Omega(1) \equiv 2 \pmod{4}$ which implies that $\Delta(5, 2) = 8$ and answers a well known open question, [33].

These results are obtained by topological methods involving careful analysis of normal data of the associated singular (weighted) 1-manifolds, see Section 2.6.4 and Figures 6 and 7. However we emphasize that the computation of relevant cohomology/bordism obstruction classes is closely related to a question of enumerating classes of immersed curves in \mathbb{R}^2 with a prescribed type and number of intersections with the coordinate axes, Figures 3 to 5, and in turn to a problem of enumerating classes of signed, cyclic A, B -words, see Example 3.8, equations (35)–(37) etc.

Along these lines one obtains other, similar equipartition results, see Proposition 3.1, Proposition 3.12 and Corollary 3.13. It is worth mentioning that some of these results have alternative proofs based on the ideal-valued, cohomological index theory. This approach has a merit that it produces fairly general and in some case quite accurate upper bounds for the function $\Delta(j, k)$, Theorem 4.2.

1.2 History of the Problem

The general problem of studying equipartitions of masses by hyperplanes, or in our reformulation the problem of determining the function $d = \Delta(j, k)$, was formulated by Branko Grünbaum in [19]. Hugo Hadwiger proved that $\Delta(2, 2) = 3$, [20], which also implies $\Delta(1, 3) = 3$. The case $k = 1$ is answered by the “ham sandwich theorem”, [9], which says that $\Delta(d, 1) = d$. Edgar Ramos, [33], building on the previous special results in [43], introduced new ideas and considerably advanced our knowledge about the function $d = \Delta(j, k)$. He showed for example that $\Delta(1, 4) \leq 5$, $\Delta(5, 2) \leq 9$, $\Delta(3, 3) \leq 9$. The *moment curve* considerations, see [33] or Section 2.4, lead to the following general lower bound

$$\Delta(j, k) \geq j(2^k - 1)/k. \quad (1)$$

As far as the general upper bounds are concerned, Ramos proved that for $j = 2^m$, one has the inequalities

$$\begin{aligned} 3j/2 \leq \Delta(j, 2) \leq 3j/2 & \quad 7j/3 \leq \Delta(j, 3) \leq 5j/2 \\ 15j/4 \leq \Delta(j, 4) \leq 9j/2 & \quad 31j/5 \leq \Delta(j, 5) \leq 15j/2 \end{aligned} \quad (2)$$

It was conjectured in [33] that the bound given by (1) is tight.

In computational geometry, the equipartition problem arose in relation to the problem of designing efficient algorithms for half-space range queries, [41]. In the meantime better partitioning techniques were found, [44], [26], [27], [28] but the complexity of the equipartition problem remains of great theoretical interest.

In combinatorial geometry, problems of partitions of sets of points and dissections of mass distributions have a long tradition and occupy one of central positions in this field. In this category are R. Rado’s “theorem for general measures” [32], B. Grünbaum’s “center point theorem” for convex bodies [19], etc. Survey articles [15], [29], [46], [47], [48] as well as the original papers [6], [33], [37], [51], cover different aspects of the equipartition problem and give a good picture of some of more recent developments in this area. For example [6] deals with partitions by k -fans with prescribed measure ratios, [37] studies equipartitions by regular wedge-like cones and the relation with the well known Knaster’s conjecture, [38] deals with general conical partitions, the “center transversal theorem” proved in [51] reveals a hidden connection between the center point theorem and the ham sandwich theorem, etc. Most of these results are obtained by topological methods. This orientation ultimately led in [49], see also [25], to the formulation of a program which unifies much of the discrete and continuous theory in the context of so called “combinatorial geometry on vector bundles”.

1.3 Mass Distributions

A continuous mass distribution, referred to in Problem 1.3 and Theorem 1.4, is a finite Borel measure μ defined by the formula $\mu(A) = \int_A f d\mu$ for an integrable density function $f : \mathbb{R}^d \rightarrow \mathbb{R}$. More generally, a mass distribution can be a Borel measure ν on \mathbb{R}^d which is a *weak limit*, [7], of a sequence ν_n of continuous measures in the sense that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} f d\nu_n = \int_{\mathbb{R}^d} f d\mu$$

for every bounded continuous function $f : \mathbb{R}^d \rightarrow \mathbb{R}$. From here one easily deduces the inequalities, $\limsup \nu_n(F) \leq \nu(F)$ for each closed set, and $\liminf \nu_n(U) \geq \nu(U)$ for each open set in \mathbb{R}^d .

Given a continuous measure/mass distribution μ , a collection of k hyperplanes is, according to Definition 1.1, an equipartition of μ if each k -orthant has the fraction $1/2^k$ of the total mass of μ . More generally, one can define an equipartition for any measure ν if the inequality $\nu(O) \leq (1/2^d)\mu(\mathbb{R}^d)$ is valid for each of 2^k open orthants determined by the collection of k -hyperplanes. This shows that Theorem 1.4 and other equipartition results which are valid for continuous measures can be easily extended to more general mass distributions. For example the statement $\Delta(7, 2) = 11$ implies that for any set $A = \{a_{ij}\}_{(i,j) \in [4] \times [7]} \subset \mathbb{R}^{11}$, a “matrix” of points in \mathbb{R}^{11} , there exist two hyperplanes H_1 and H_2 such

that each of the associated open quadrants contains at most one point from each of the 7 columns of the matrix A .

In this paper we restrict our attention to measures which are weak limits of continuous measures which appears to be sufficient for all combinatorial applications. Among the interesting examples are measurable sets, counting measures, k -dimensional Hausdorff measures etc.

2 Proof Technique

In this section we collect most of the results needed for the proof of Theorem 1.4 and other related equipartition results. For the readers convenience, in the first subsection we outline the general proof scheme while other subsections can be seen as an elaboration of some of the ideas used in individual steps.

2.1 The General Scheme of the Proof

Here is a general proof scheme that we follow in this paper and which in principle can be, with necessary modifications, applied to many problems about (equi)partitions of masses.

1. The equipartition problem, Problem 1.3, is reduced to the question of (non)existence of an equivariant map or equivalently, to the question of the (non)existence of a nowhere zero, continuous cross-section of a vector bundle. This is a topological problem.
2. The topological problem is reduced to the question of computing relevant (co)homology, or G -bordism obstruction classes.
3. The obstruction classes belong to the associated (co)homology or bordism groups. The computation of these groups usually involves a mixture of topological and algebraic ideas.
4. The obstruction class can be, under mild conditions, computed/identified by a careful analysis of the equipartition problem for a special, sufficiently “generic” collection of measures. Our primary choice are uniform/interval measures distributed along the moment curve $\Gamma_d = \{t, t^2, \dots, t^n \mid t \in R\} \subset \mathbb{R}^d$. A geometric problem arises, involving the analysis of possible ways a curve, carrying collections of intervals, can be immersed in the coordinate two plane.
5. The analysis of the solution set of all equipartitions for the special choice of measures, linked with the problem of immersions of curves in the previous step, leads to a problem of enumeration of classes of circular words in alphabet $\{A, B\}$. This is a problem of enumerative combinatorics typically solved by Polyá enumeration or Möbius inversion technique.
6. The relevant obstruction class is linked in Step (5) with a function which has a clear arithmetical/combinatorial meaning which eventually leads to Theorem 1.4.

2.2 An Approach Based on G -bordism

Here is an example how one can approach an equipartition problem, in a direct and geometrically transparent fashion, via equivariant bordism. This is essentially the approach of this paper up to some technical refinements or detours involving equivariant maps and equivariant cohomology.

Suppose we want to prove that $\Delta(1, 2) = 2$ i.e. that for each measurable set $A \subset \mathbb{R}^2$, there exist two lines L_1 and L_2 which form a “coordinate system” so that each quadrant contains a quarter of the measure of A . The problem itself is of course quite elementary and we use it here merely to demonstrate the general ideas.

The *configuration space* of all candidates for the solution is the space of all ordered pairs of oriented lines in \mathbb{R}^2 . If we fix an embedding $\mathbb{R}^2 \hookrightarrow \mathbb{R}^3$ such that \mathbb{R}^2 does not contain the origin in \mathbb{R}^3 , an oriented line L in \mathbb{R}^2 is seen as an intersection of \mathbb{R}^2 with a unique oriented, central plane P in \mathbb{R}^3 . So the compactified configuration space is the manifold $M = S^2 \times S^2$ of all pairs of oriented, central 2-planes in \mathbb{R}^3 . We note that dihedral group $G := \mathbb{D}_8$ arises naturally in this context as a group of all symmetries of a pair of oriented lines (planes).

For a *generic* measurable set A , the collection of all pairs $(L_1, L_2) \in M$ which form an equipartition for A is a 1-dimensional G -manifold M_A . For example if A is a unit disc D , the solution set M_D is a union of 4 circles. Here we do not make precise what is meant by a generic measure. Instead we naively assume, for the sake of this example, that there exists such a notion of genericity for measurable sets/measures so that each measurable set A can be well approximated by generic measures. Moreover, we assume that for any two measurable sets A and B there exists a path of generic measures μ_t , $t \in [0, 1]$, so that μ_0 is an approximation of A , μ_1 is an approximation for B and the solution set

$$M_{\{\mu_t\}_{t \in [0,1]}} := \{(L_1, L_2; t) \mid (L_1, L_2) \text{ is an equipartition for } \mu_t\} \subset S^2 \times S^2 \times [0, 1] \quad (3)$$

is a 2-dimensional manifold (bordism) connecting solution sets for measures μ_0 and μ_1 . The group $\Omega_1(\mathbb{D}_8)$ of classes of 1-dimensional, free \mathbb{D}_8 -manifolds is found to be isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, see Section 2.6.3, and the \mathbb{D}_8 -solution manifold M_D , associated to the unit disc in \mathbb{R}^2 , is shown to represent a nontrivial element in this group.

It immediately follows that for any measurable set $A \subset \mathbb{R}^2$, the solution set M_A is nonempty. Indeed, suppose $M_A = \emptyset$. Let $\{\mu_t\}_{t \in [0,1]}$ be a path of generic measures such that μ_0 approximates A and μ_1 approximates D . If these approximations are sufficiently good, we deduce that the solution set M_{μ_0} is empty and that $[M_{\mu_1}]$ and $[M_D]$ represent the same element in $\Omega_1(\mathbb{D}_8)$. This is a contradiction since $M_{\mu_1} = \partial(M)$ where $M = M_{\{\mu_t\}_{t \in [0,1]}}$, i.e. M_D would represent a trivial element in $\Omega_1(\mathbb{D}_8)$.

Remark 2.1 It is worth noting that the scheme outlined above, if applicable, shows that a general equipartition problem can be solved by a careful analysis of the solution set of a well chosen, particular measure/measurable set, the unit disc D in our example above. Unfortunately, in higher dimensions the unit balls do not represent generic measures, i.e. their solution manifolds are very special and cannot be used for the evaluation of the relevant obstruction elements in $\Omega_*(\mathbb{D}_8)$. Instead one uses uniform, interval measures on the *moment curve*, see Section 2.4. We note also that although the program outlined above can actually be carried on, we will use a more standard and equivalent approach via obstruction classes and equivariant Poincaré duality, Section 2.3. Nevertheless, the idea of a generic measure is sufficiently interesting and attractive in itself and we hope to return to it in a later paper.

2.3 Equivariant Maps

It is often more convenient to work with equivariant maps and their zero sets rather than with measures and their solution manifolds. Each measure μ leads naturally to an equivariant map $A_\mu : M \rightarrow V$, where M is a free G -manifold and V a G -representation. Working with G -equivariant maps often yields more general results and it is sometimes technically more convenient. For example one easily constructs G -maps which are transversal to G -submanifolds of V , thus avoiding the question of “generic” measures from Section 2.2.

2.3.1 The central paradigm

The *configuration space/test map* paradigm, [47], is apparently one of the central ideas for applications of topological methods in geometric and discrete combinatorics. Review papers, [4], [5], [8], [46], [47], [48] give a detailed picture of the genesis of these ideas and emphasize their role in the solutions of well known combinatorial problems like Kneser’s conjecture (L. Lovasz, [24]), “splitting necklace problem” (N. Alon, [3], [4]), Colored Tverberg problem (R. Živaljević, S. Vrećica, [52], [39]) etc.

The idea can be outlined as follows. One starts with a configuration space or manifold $M_{\mathcal{P}}$ of all candidates for the solution of a geometric/combinatorial problem \mathcal{P} . For example, the equipartition problem \mathcal{P} led in Section 2.2 to the configuration space $M_{\mathcal{P}} \cong S^2 \times S^2$ of all pairs of oriented planes in \mathbb{R}^3 . The following step is a construction of a test map $f : M_{\mathcal{P}} \rightarrow V_{\mathcal{P}}$ which measures how far is a given candidate configuration from being a solution. More precisely, there is a subspace Z of the test space $V_{\mathcal{P}}$ such that a configuration $C \in M_{\mathcal{P}}$ is a solution if and only if $f(C) \in Z$. The inner symmetries of the problem \mathcal{P} typically show up at this stage. This means that there is a group G of symmetries of $X_{\mathcal{P}}$ which acts on $V_{\mathcal{P}}$, such that Z is a G -invariant subspace of $V_{\mathcal{P}}$, which turns $f : M_{\mathcal{P}} \rightarrow V_{\mathcal{P}}$ into an equivariant map. If a configuration C with the desired property $f(C) \in Z$ does not exist, then there arises an equivariant map $f : M_{\mathcal{P}} \rightarrow V_{\mathcal{P}} \setminus Z$. The final step is to show by topological methods that such a map does not exist.

2.3.2 Equipartition problem revisited

Our first choice for the configuration space suitable for the equipartition problem (Problem 1.3) is the manifold of all ordered collections $C = (H_1, \dots, H_k)$ of oriented hyperplanes in \mathbb{R}^d . In order to obtain a compact manifold, we move one dimension up and embed \mathbb{R}^d in \mathbb{R}^{d+1} , say as the hyperplane determined by the equation $x_{d+1} = 1$. Then each oriented hyperplane H in $\mathbb{R}^d \cong \{x \in \mathbb{R}^{d+1} \mid x_{d+1} = 1\}$ is obtained as an intersection $H = \mathbb{R}^d \cap H'$, where H' is a uniquely defined, oriented, $(d+1)$ -dimensional subspace of \mathbb{R}^{d+1} . The oriented subspace H' is determined by the corresponding orthogonal unit vector $u \in S^d \subset \mathbb{R}^{d+1}$, so the natural environment for collections $C = (H_1, \dots, H_k)$, and our actual choice for the configuration space is $M_{\mathcal{P}} := (S^d)^k$. The group which acts on the configuration manifold $M_{\mathcal{P}}$ is the reflection group $W_k := (\mathbb{Z}/2)^k \rtimes S_k$. The test space $V = V_{\mathcal{P}}$ for a single measure is defined as follows. Let μ be a measure defined on \mathbb{R}^d and let μ' be the measure induced on \mathbb{R}^{d+1} by the embedding $\mathbb{R}^d \hookrightarrow \mathbb{R}^{d+1}$. A k -tuple $(u_1, \dots, u_k) \in (S^d)^k$ of unit vectors determines a k -tuple $C = (H'_1, \dots, H'_k)$ of oriented $(d+1)$ -dimensional subspaces of \mathbb{R}^{d+1} . The k -tuple C divides \mathbb{R}^{d+1} into 2^k -orthants Ort_{β} which are naturally indexed by 0-1 vectors $\beta = \beta_C \in \mathbb{F}_2^k$. Let $b_{\beta} : M_{\mathcal{P}} \rightarrow \mathbb{R}$ be the function defined by $b_{\beta}(C) := \mu'(\text{Ort}_{\beta}) = \mu(\text{Ort}_{\beta} \cap \mathbb{R}^d)$. Let $B_{\mu} : (S^d)^k \rightarrow \mathbb{R}^{2^k}$ be the function defined by $B_{\mu}(C) = (b_{\beta}(C))_{\beta \in \mathbb{F}_2^k}$. The μ -test space $V_k = V_{\mathcal{P}} \cong \mathbb{R}^{2^k}$ has a natural action of the group $W_k := (\mathbb{Z}/2)^k \rtimes S_k$ such that the map B_{μ} is W_k -equivariant. The W_k -representation V_k , restricted to the subgroup $(\mathbb{Z}/2)^k \hookrightarrow W_k$, reduces to the regular representation $\text{Reg}((\mathbb{Z}/2)^k)$ of the group $(\mathbb{Z}/2)^k$. The “zero” subspace $Z_{\mathcal{P}}$ is defined as the trivial, 1-dimensional W_k -representation V_k^0 contained in V_k . Let U_k be the complementary W_k -representation, $U_k \cong V_k/V_k^0$ and $A_{\mu} : (S^d)^k \rightarrow U_k$ the induced, W_k -equivariant map. By the construction we have the following proposition which says that A_{μ} is a genuine test map for the μ -equipartition problem.

Proposition 2.2 *A k -tuple $C = (H_1, \dots, H_k) \in M_{\mathcal{P}} = (S^d)^k$ of oriented hyperplanes is an equipartition of a measure μ defined on \mathbb{R}^d if and only if $A_{\mu}(C) = 0$. Similarly, given a collection $\{\mu_1, \dots, \mu_j\}$ of j measures in \mathbb{R}^d , a k -tuple C is an equipartition for each of the measures μ_i if and only if $A(C) = 0$ where $A : (S^d)^k \rightarrow U_k^{\oplus j}$ is the W_k -equivariant map defined by $A(C) := (A_{\mu_i})_{i=1}^j$.*

This proposition motivates the following problem.

Problem 2.3 *Determine or find a nontrivial estimate for the minimum integer $d := \Theta(j, k)$ such that there does not exist an W_k -equivariant map*

$$A : (S^d)^k \rightarrow S(U_k^{\oplus j}) \quad (4)$$

where U_k is the W_k -representation described above, and $S(U_k^{\oplus j})$ is the W_k -invariant (unit) sphere in the representation $U_k^{\oplus j}$.

Clearly Proposition 2.2, which says that the inequality $\Theta(j, k) \geq \Delta(j, k)$ always holds, provides a tool for proving equipartition results for measures in \mathbb{R}^d .

2.3.3 Equivariant Poincaré duality

Once a problem is reduced to the question of (non)existence of equivariant map, one can use some standard topological tools for its solution. For example one can use the cohomological index theory for this purpose, [17], [14], [47], [48]. This approach is discussed in Section 4.1. In this paper our main tool is elementary equivariant obstruction theory [13], refined by some basic equivariant bordism, and group homology calculations.

Suppose that M^n is orientable, n -dimensional, free G -manifold and that V is a m -dimensional, real representation of G . Then the first obstruction for the existence of an equivariant map $f : M \rightarrow S(V)$, is a cohomology class

$$\omega \in H^m(M, \pi_{m-1}(S(V)))$$

in the appropriate equivariant cohomology group, where $\pi_k(S(V))$ is seen as G -module. The action of G on M induces a G -module structure on the group $H_n(M, \mathbb{Z}) \cong \mathbb{Z}$ which is denoted by \mathcal{Z} . The associated homomorphism $\theta : G \rightarrow \{-1, +1\}$ is called the orientation character. Let A be a (left) G -module. The Poincaré duality for equivariant (co)homology is the following isomorphism, [40],

$$H_G^k(M, A) \xrightarrow{D} H_{n-k}^G(M, A \otimes \mathcal{Z}) \quad (5)$$

Moreover, if $g : M \rightarrow V$ is a smooth map transversal to $\{0\} \subset V$, then $V := g^{-1}(0)$ is an oriented G -submanifold of M and the dual $D(\omega)$ of the obstruction class ω is represented by the fundamental class $[V]$ of V .

As an illustration, we compute the coefficient G -module $N := A \otimes \mathcal{Z}$ in the case of interest in this paper. According to Section 2.3.2, Problem 2.3, the equipartition problem can be reduced to the question of existence of an equivariant map

$$A : (S^d)^k \rightarrow S(U_k^{\oplus j}). \quad (6)$$

Although the computation in full generality is not much more difficult, we restrict our attention to the case $k = 2$, the only case which is systematically studied in this paper. The group $G \cong W_2$, turns out to be a dihedral group \mathbb{D}_8 of order 8. We work with the presentation of $G \cong \mathbb{D}_8$ described in Section 2.6.1. The equivariant map (6) for $k = 2$ has the form $A : S^d \times S^d \rightarrow U_2^{\oplus j}$. The 3-dimensional vector space U_2 can be decomposed, as a \mathbb{D}_8 -representation, into a direct sum $U_2 \cong E_1 \oplus E_2$, where E_2 is the standard, 2-dimensional \mathbb{D}_8 -representation, and E_1 a 1-dimensional representation where α and β act non trivially while the action of γ is trivial. Then the \mathbb{D}_8 -structure on the group $A \otimes \mathcal{Z} \cong \mathbb{Z}$ can be read off the following table.

	α	β	γ
$H_{2d}(S^d \times S^d)$	$(-1)^{d+1}$	$(-1)^{d+1}$	$(-1)^{d^2}$
U_2	$+1$	$+1$	-1
$U_2^{\oplus j}$	$+1$	$+1$	$(-1)^j$
$A \otimes \mathcal{Z}$	$(-1)^{d+1}$	$(-1)^{d+1}$	$(-1)^{d+j}$

In Section 3.2, we will be particularly interested in the case $\Delta = 2d - 3j = 1$, i.e. in the tuples $(d, j) = (3m + 2, 2m + 1)$, where m is a non negative integer. Then the last row of the table describing the \mathbb{D}_8 -module structure on the coefficient group N has the form

$$N = A \otimes \mathcal{Z} \mid (-1)^{m+1} \mid (-1)^{m+1} \mid (-1)^{m+1}$$

If m is odd, N is a trivial \mathbb{D}_8 -module which is denoted simply by \mathbb{Z} . If m is even, then N is a group isomorphic to \mathbb{Z} while all the generators α, β, γ act nontrivially. This module structure is in Section 2.6.2 denoted by \mathcal{Z} .

2.4 Moment Curve

The moment curve $\Gamma_d = \{(t, t^2, \dots, t^d) \mid t \in R\}$ and the closely related Carathéodory curve $C_n = \{(\cos t, \sin t, \cos 2t, \sin 2t, \dots, \cos nt, \sin nt) \mid t \in [0, 2\pi]\}$, have numerous applications in geometric combinatorics, [2], [45], [50]. The key property of these curves is that each hyperplane H intersects Γ_g in at most d points; respectively $2n$ points in the case of Carathéodory curve C_n . Let I_1, I_2, \dots, I_j be a collection of disjoint intervals on the moment curve Γ_d , $I_i = [a_i, b_i]$, $a_1 < b_1 < a_2 < b_2 \dots a_j < b_j$. Let μ_i be the uniform probability measure on I_i and $\hat{\mu}_i$ the induced measure on \mathbb{R}^d , $\hat{\mu}_i(B) := \mu_i(B \cap I_i)$. A collection $\mathcal{H} = \{H_1, H_2, \dots, H_k\}$ of k hyperplanes in \mathbb{R}^d can have at most kd intersection points with the curve Γ_d . If \mathcal{H} is an equipartition of all measures $\hat{\mu}_i$, then the number of intersection points is at least $j(2^k - 1)$. It follows that if an equipartition exists then $kd \geq j(2^k - 1)$ and we obtain the following lower bound for the function $d = \Delta(j, k)$.

Proposition 2.4 ([33])

$$\Delta(j, k) \geq j(2^k - 1)/k \quad (7)$$

2.5 Circular $\{A, B\}$ -words

Let \mathcal{A}_n be the set of all words of length $2n$ in the alphabet $\{A, B\}$ and let \mathcal{R}_n be the subset of all words with the same number of occurrences of letters A and B . These words will be occasionally referred to as *balanced* words.

Definition 2.5 Given a word $w = x_1 x_2 \dots x_{2n}$ in \mathcal{A}_n , let $C(w) := x_2 x_3 \dots x_{2n} x_1$ be its cyclic permutation. The conjugation $*$: $\mathcal{A}_n \rightarrow \mathcal{A}_n$ is inductively defined by $A^* = B$, $B^* = A$, $(uv)^* = u^* v^*$, i.e. the conjugation is an involution on \mathcal{A}_n which replaces each occurrence of a letter A in w , by a letter B and vice versa. The operator C is a generator of a $\mathbb{Z}/2n$ -action on \mathcal{A}_n while C and $*$ together generate a $G := \mathbb{Z}/2n \times \mathbb{Z}/2$ action on both \mathcal{A}_n and the set \mathcal{R}_n of balanced words.

Definition 2.6 A circular word is a word in \mathcal{A}_n up to a cyclic permutation. More precisely, a circular word is a $(\mathbb{Z}/2n)$ -orbit in the $(\mathbb{Z}/2n)$ -set \mathcal{A}_n . If $w \in \mathcal{A}_n$, then the associated circular word in $\mathcal{A}_n/(\mathbb{Z}/2n)$ is denoted by $[w]$. Let $R(n) := |\mathcal{R}_n/(\mathbb{Z}/2n)|$ be the number of balanced, circular words of length $2n$.

Definition 2.7 If $w \in \mathcal{A}_n$, let $\text{Per}_{\mathbb{Z}/2n}(w) := \min\{l \mid C^l(w) = w\}$. The number $p = \text{Per}_{\mathbb{Z}/2n}(w)$ is called the period of w and w is referred to as a word of primitive period p . A word $w \in \mathcal{A}_m$ of primitive period $2m$ is called primitive. If $w \in \mathcal{A}_m$ is primitive, then the associate circular word $[w]$ is also called primitive. Let $P(m)$, respectively $Q(m)$, be the number of primitive, circular, words of length m , respectively the number of primitive, circular, balanced words of length $2m$.

Polyá's enumeration theory, [18], [23], deals with the problem of enumerating the G -orbits of classes of weighted words/functions. An initial example is the following formula for the number $R(n)$ of, balanced, circular $\{A, B\}$ -words

$$R(n) = \frac{1}{2n} \sum_{m|n} \binom{2m}{m} \phi(n/m) \quad (8)$$

where ϕ is Euler totient function. An alternative approach to the problem of counting circular words is via Möbius inversion theorem, [1], [18], [34]. The set \mathcal{R}_m of all, balanced words of length $2m$ is a disjoint union of words of primitive period $2k$ for some divisor k of m . Hence,

$$\binom{2m}{m} = \sum_{k|m} (2k) Q(k) \quad (9)$$

and the Möbius inversion yields the following equation

$$Q(m) = \frac{1}{2m} \sum_{k|m} \binom{2k}{k} \mu(m/k). \quad (10)$$

Similarly,

$$2^m = \sum_{k|m} kP(k) \quad \text{yields} \quad P(m) = \frac{1}{m} \sum_{k|m} 2^k \mu(m/k) \quad (11)$$

Keeping in mind the connection between self-conjugated, balanced, circular AB -words with the bordism obstruction classes $o \in \Omega_1(\mathbb{D}_8)$ established in Section 3, we now focus our attention on the action of the involution $*$ on the set $\mathcal{R}_n/(\mathbb{Z}/2n)$ of circular words.

Definition 2.8 A word $w \in \mathcal{R}_n$ is special if $C^r(w) = w^*$ for some r . A word w is special iff the associated circular word $[w] \in \mathcal{R}_n/(\mathbb{Z}/2n)$ is self-conjugated in the sense that $*([w]) = [w]$. A primitive, special word in \mathcal{R}_m is called $*$ -primitive. Let $A(m)$ be the number of all $*$ -primitive circular words in \mathcal{R}_m . A circular, special word is also called self-conjugated.

Lemma 2.9 A word $w \in \mathcal{R}_n$ is special if and only if it has the form $(aa^*)(aa^*) \dots (aa^*)$ for some a . This representation is referred to as a special representation of the word w . A word is $*$ -primitive if it has a unique special representation of the form $w = bb^*$.

It follows from Lemma 2.9 that the number of circular, self-conjugated words in $\mathcal{R}_n/(\mathbb{Z}/2n)$ of primitive period $2m$, where $m|n$, is also $A(m)$.

In light of applications in Section 3, it would be interesting to have an explicit, simple formula for the function $A(m)$. This problem is certainly amenable to more refined methods from Polyá enumeration/Möbius inversion theory, see [34] or [23], Section II.5. However, for our purposes it is sufficient to determine the residuum of $A(m)$ modulo 2.

Proposition 2.10

$$A(m) \equiv P(2m) \pmod{2}$$

Proof: Let $\mathcal{A}_n(m)$ be the set of all circular words in $\mathcal{A}_n/(\mathbb{Z}/2n)$ of primitive period $2m$ and let $\mathcal{A}_n^*(m)$ be its subset of self-conjugated circular words. For the proof of the proposition, it is sufficient to observe that $|\mathcal{A}_n(m)|P(2m)$, $|\mathcal{A}_n^*(m)| = A(m)$ and that $\mathcal{A}_n^*(m)$ is the fixed point set for the involution $*$: $\mathcal{A}_n(m) \rightarrow \mathcal{A}_n(m)$. \square

Proposition 2.11 $P(2k) \equiv_2 1$ if either k is odd, square-free integer or $k = 2q$ where q is odd, square-free integer. Otherwise, $P(2k) \equiv_2 0$.

Proof: The proof follows from the explicit formula for the function $P(m)$ given in equation (11) and well known properties of the Möbius function. \square

2.6 Homology and Bordism Computations

By equivariant Poincaré duality, Section 2.3.3, the dual $D(\omega)$ of the first obstruction cohomology class $\omega \in H_G^m(M, \pi_{m-1}S(V))$ lies in the group $H_{n-m}^G(M, \pi_{m-1}S(V) \otimes \mathcal{Z})$. If M is $(n-m)$ -connected, then there is an isomorphism ([11], Theorem II.5.2)

$$H_{n-m}^G(M, \pi_{m-1}S(V) \otimes \mathcal{Z}) \xrightarrow{\cong} H_{n-m}(G, \pi_{m-1}S(V) \otimes \mathcal{Z}).$$

This allows us to interpret $D(\omega)$ as an element in the letter group. Moreover, if the coefficient G -module $\pi_{m-1}S(V) \otimes \mathcal{Z}$ is trivial, then the homology group $H_{n-m}(G, \mathbb{Z}) \cong H_{n-m}(BG, \mathbb{Z})$ is for $n-m \leq 3$ isomorphic, [12], to the oriented G -bordism group $\Omega_{n-m}(G) \cong \Omega_{n-m}(BG)$.

Our objective is to identify the relevant obstruction classes. Already the algebraically trivial case $H_0(G, M) \cong M_G$, where $M_G = \mathbb{Z} \otimes_G M$ is the group of coinvariants, may be combinatorially sufficiently interesting. However, the most interesting examples explored in this paper involve the identification of 1-dimensional obstruction classes. Since these classes in practice usually arise as the fundamental classes of zero set manifolds, our first choice will be the bordism group $\Omega_1(G)$.

2.6.1 Dihedral group \mathbb{D}_8

In this section we focus our attention on the dihedral group $G = W_2 = \mathbb{D}_8$. As usual, the group $\mathbb{D}_8 \cong (\mathbb{Z}/2)^{\oplus 2} \rtimes \mathbb{Z}/2$ is identified to the group of all symmetries of a square with the generators α, β of $(\mathbb{Z}/2)^{\oplus 2}$ seen as reflections with respect to the x -axes and y -axes respectively, while γ is the reflection in the line $x = y$.

Low dimensional homology groups are usually not difficult to compute, say via Hochschild-Serre spectral sequence. However, in our applications we need more precise information about the representing cycles of these groups, and this is the reason why work directly with chains and resolutions.

A convenient $Z[G]$ -resolution of Z for the group $G = \mathbb{D}_8 = (\mathbb{Z}/2)^{\oplus 2} \rtimes \mathbb{Z}/2$, or more geometrically a convenient EG -space with an economical G -CW-structure, can be described as follows.

Let $EG := S^\infty \times S^\infty \times S^\infty$. The action of G on EG is described as follows

$$\begin{aligned}\alpha(x, y, z) &= (-x, y, z) \\ \beta(x, y, z) &= (x, -y, z) \\ \gamma(x, y, z) &= (y, x, -z)\end{aligned}\tag{12}$$

A presentation of \mathbb{D}_8 as the group freely generated by the generators α, β, γ subject to the relations

$$\begin{aligned}\alpha^2 &= \beta^2 = \gamma^2 = 1 \\ \alpha\beta &= \beta\alpha \\ \alpha\gamma &= \gamma\beta \\ \beta\gamma &= \gamma\alpha\end{aligned}\tag{13}$$

shows that the action described by (12) is well defined. There is a natural G -invariant CW -structure on $EG = (S^\infty)^{\times 3}$ which is described as the product of usual $\mathbb{Z}/2$ -invariant CW -structures on S^∞ . In more details, let

$$\dots \xrightarrow{1-t} Z[\mathbb{Z}/2]x_2 \xrightarrow{1+t} Z[\mathbb{Z}/2]x_1 \xrightarrow{1-t} Z[\mathbb{Z}/2]x_0 \longrightarrow 0\tag{14}$$

be the usual $\mathbb{Z}/2$ -invariant cellular chain complex of S^∞ with one cell x_i in each dimension. Then the cellular chain complex $\mathcal{C}_G = (\{C_n\}_{n \geq 0}, \partial)$ for EG can be seen as a tensor product of cellular chain complexes of individual spheres with t in (14) replaced in the corresponding sphere by α, β , and γ respectively, and the corresponding generators/cells x_i are denoted respectively by a_i, b_i, c_i . So, a typical n -cell in \mathcal{C}_G has the form $g(a_i \times b_j \times c_k)$ or in a more algebraic fashion $g(a_i \otimes b_j \otimes c_k)$, for some $g \in G$ where $i + j + k = n$. Note that according to (12), the action of G on \mathcal{C}_G , the cellular chain complex of EG , is described by the equalities

$$\begin{aligned}\alpha(a_i \otimes b_j \otimes c_k) &= (\alpha a_i \otimes b_j \otimes c_k) \\ \beta(a_i \otimes b_j \otimes c_k) &= (a_i \otimes \beta b_j \otimes c_k) \\ \gamma(a_i \otimes b_j \otimes c_k) &= (a_j \otimes b_i \otimes \gamma c_k)\end{aligned}\tag{15}$$

We are interested in the first homology groups $H_1(G, M)$ where M is either Z seen as a trivial, right G -module, or $M = \mathcal{Z}$, where $\mathcal{Z} \cong Z$ as a group, while the G -action on \mathcal{Z} is defined by

$$t\alpha = t\beta = t\gamma = -t$$

where $t \in \mathcal{Z}$ is a generator.

2.6.2 Homology computations

The group $H_n(G, M)$ is by definition the appropriate homology group of the following chain complex.

$$\dots \xrightarrow{\partial} M \otimes_G C_2 \xrightarrow{\partial} M \otimes_G C_1 \xrightarrow{\partial} M \otimes_G C_0 \xrightarrow{\partial} 0\tag{16}$$

In order to simplify the notation, we will often use the abbreviation (a_i, b_j, c_k) for $t \otimes_G (a_i \otimes b_j \otimes c_k) \in M \otimes_G C_n$, where $n = i + j + k$.

Case 1 $M = \mathbb{Z}$

The following equalities are easily derived from (15).

$$\begin{aligned}\partial(a_1, b_0, c_0) &= (a_0 - \alpha a_0, b_0, c_0) = 0 \\ \partial(a_0, b_1, c_0) &= (a_0, b_0 - \beta b_0, c_0) = 0 \\ \partial(a_0, b_0, c_1) &= (a_0, b_0, c_0 - \gamma c_0) = 0\end{aligned}\tag{17}$$

$$\begin{aligned}\partial(a_2, b_0, c_0) &= 2(a_1, b_0, c_0) \\ \partial(a_0, b_2, c_0) &= 2(a_0, b_1, c_0) \\ \partial(a_0, b_0, c_2) &= 2(a_0, b_0, c_1)\end{aligned}\tag{18}$$

$$\begin{aligned}\partial(a_1, b_1, c_0) &= (\partial a_1, b_1, c_0) - (a_1, \partial b_1, c_0) = 0 \\ \partial(a_1, b_0, c_1) &= -(a_1, b_0, c_0) + (a_0, b_1, c_0) \\ \partial(a_0, b_1, c_1) &= -(a_1, b_0, c_0) + (a_0, b_1, c_0)\end{aligned}\tag{19}$$

From these calculations we easily reach the following conclusion.

Proposition 2.12

$$H_1((\mathbb{Z}/2)^{\oplus 2} \rtimes \mathbb{Z}/2, \mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

The generators are $X = (a_1, b_0, c_0)$ and $Y = (a_0, b_0, c_1)$.

Case 2 $M = \mathbb{Z}$

As before, the calculations are based on equations in (15).

$$\begin{aligned}\partial(a_1, b_0, c_0) &= (a_0 - \alpha a_0, b_0, c_0) = 2(a_0, b_0, c_0) \\ \partial(a_0, b_1, c_0) &= (a_0, b_0 - \beta b_0, c_0) = 2(a_0, b_0, c_0) \\ \partial(a_0, b_0, c_1) &= (a_0, b_0, c_0 - \gamma c_0) = 2(a_0, b_0, c_0)\end{aligned}\tag{20}$$

$$\begin{aligned}\partial(a_2, b_0, c_0) &= (a_1 + \alpha a_1, b_0, c_0) = 0 \\ \partial(a_0, b_2, c_0) &= (a_0, b_1 + \beta b_1, c_0) = 0 \\ \partial(a_0, b_0, c_2) &= 2(a_0, b_0, c_1 + \gamma c_1) = 0\end{aligned}\tag{21}$$

$$\begin{aligned}\partial(a_1, b_1, c_0) &= (a_0 - \alpha a_0, b_1, c_0) - (a_1, b_0 - \beta b_0, c_0) \\ &= 2(a_0, b_1, c_0) - 2(a_1, b_0, c_0) \\ \partial(a_1, b_0, c_1) &= (a_0 - \alpha a_0, b_0, c_1) - (a_1, b_0, c_0 - \gamma c_0) \\ &= 2(a_0, b_0, c_1) - (a_1, b_0, c_0) - (a_0, b_1, c_0) \\ \partial(a_0, b_1, c_1) &= (a_0, b_0 - \beta b_0, c_1) - (a_0, b_1, c_0 - \gamma c_0) \\ &= 2(a_0, b_0, c_1) - (a_0, b_1, c_0) - (a_1, b_0, c_0)\end{aligned}\tag{22}$$

As a consequence we have the following proposition.

Proposition 2.13

$$H_1((\mathbb{Z}/2)^{\oplus 2} \rtimes \mathbb{Z}/2, \mathbb{Z}) \cong \mathbb{Z}/4$$

Proof: It follows from (20) that the elements

$$\begin{aligned}X_{ab} &:= (a_1, b_0, c_0) - (a_0, b_1, c_0) \\ X_{bc} &:= (a_0, b_1, c_0) - (a_0, b_0, c_1) \\ X_{ca} &:= (a_0, b_0, c_1) - (a_1, b_0, c_0)\end{aligned}\tag{23}$$

generate the group of cycles. Moreover, essentially the only relation among them is

$$X_{ab} + X_{bc} + X_{ca} = 0. \quad (24)$$

The relations (22), expressed in terms of X_{ab}, X_{bc}, X_{ca} , read as follows

$$2X_{ab} = 0 \quad X_{ca} = X_{bc}. \quad (25)$$

It immediately follows that the homology group $H_1((\mathbb{Z}/2)^{\oplus 2} \rtimes \mathbb{Z}/2, \mathcal{Z})$ is isomorphic to $\mathbb{Z}/4$ with elements $X_{ca} = X_{bc}$ representing a generator. \square

2.6.3 Geometric interpretation

The analysis from the previous section allows us to give a precise geometric interpretation of elements of the groups $H_1(\mathbb{D}_8, \mathbb{Z}) \cong \Omega_1(\mathbb{D}_8) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and $H_1(\mathbb{D}_8, \mathcal{Z}) \cong \mathbb{Z}/4$. By definition $\Omega_1(\mathbb{D}_8)$ is the equivariant bordism group of all 1-dimensional, oriented, free \mathbb{D}_8 -manifolds. Minimal examples are manifolds of the form $\mathbb{D}_8 \times_{\Gamma} S^1 =: M_{\Gamma}$, where Γ is a subgroup of \mathbb{D}_8 which acts freely on S^1 . The corresponding class in $\Omega_1(\mathbb{D}_8)$ is denoted by $[M_{\Gamma}]$. Γ is either the trivial group, a cyclic group of order 2 or a cyclic group of order 4. In order to associate these manifolds to the elements X, Y and $Z := X + Y$ of the group $H_1(\mathbb{D}_8, \mathbb{Z})$, described in Proposition 2.12, let us inspect the diagram depicted on the Figure 1 (A). Both diagrams (A) and (B) represent a 3-dimensional torus $T^3 = S^1 \times S^1 \times S^1$, viewed as a \mathbb{D}_8 -subcomplex of the complex $E\mathbb{D}_8 = S^{\infty} \times S^{\infty} \times S^{\infty}$ described in Section 2.6.1. Note that the whole 1-skeleton and a part of 2-skeleton of $E\mathbb{D}_8$ are included in this complex. This implies that all 1-cycles should be visible in this picture. For example in Figure 1 (A), the 1-cell (a_1, b_0, c_0) , associated to the generator X , is represented by the vector $P_1 Q_1$. Then $P_1 Q_1 + \alpha P_1 Q_1 = P_1 Q_1 + P_2 Q_2$ is a 1-chain which determines a circle C_1 invariant by the group $\mathbb{Z}/2 = \{1, \alpha\}$. The whole \mathbb{D}_8 -orbit of $P_1 Q_1$ (respectively C_1) is in Figure 1 (A) represented by the union of four circles easily identified as the manifold $M_{(\alpha)}$ where (α) is the subgroup of \mathbb{D}_8 generated by α . A similar analysis shows that $Y = (a_0, b_0, c_1)$ is represented by $M_{(\gamma)}$. Note that conjugated groups Γ_1 and Γ_2 determine manifolds M_{Γ_1} and M_{Γ_2} which are isomorphic as \mathbb{D}_8 -manifolds and represent the same elements in $\Omega_1(\mathbb{D}_8)$.

The following proposition summarizes the results about the geometric representatives of elements of the group $H_1(\mathbb{D}_8, \mathbb{Z})$.

Proposition 2.14 *The nontrivial elements in $H_1(\mathbb{D}_8, \mathbb{Z}) \cong \Omega_1(\mathbb{D}_8) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ are*

$$X = [M_{(\alpha)}] = [M_{(\beta)}], \quad Y = [M_{(\gamma)}] = [M_{(\alpha\beta\gamma)}], \quad Z = [M_{(\alpha\gamma)}] = [M_{(\beta\gamma)}] \quad (26)$$

while the trivial element in $\Omega_1(\mathbb{D}_8)$ is represented by 1-manifolds $M_{(e)}$ and $M_{(\alpha\beta)}$.

Proof: As in the special case of the generator X , one starts with a “vector description” $P_1 Q_1$ of the corresponding chain in the torus T^3 , Figure 1 (A). The smallest \mathbb{D}_8 -invariant chain containing $P_1 Q_1$ yields the associated manifold M_{Γ} . \square

A similar geometric interpretation is possible for the elements of the group $H_1(\mathbb{D}_8, \mathcal{Z}) \cong \mathbb{Z}/4$. This time however we have to be more careful. As a motivation we should keep in mind that the representatives of this group appear “in nature” as *weighted* \mathbb{D}_8 -invariant 1-manifolds N where the weights take into account the orientation of the normal bundle $\nu(N)$, seen as a small tubular neighborhood of N in $M = (S^d)^k$. Recall that we already met weighted chains $t \otimes_G (a_i \otimes b_j \otimes c_k)$ in Section 2.6.2. Let M be a free, \mathbb{D}_8 -invariant, oriented, 1-manifold $M = M_{\Gamma} = \mathbb{D}_8 \times_{\Gamma} S^1 = \cup_{g \in G/\Gamma} S_g^1$. In other words M is represented as a disjoint union of oriented circles S_g^1 where the indices g run over the representatives of the corresponding cosets. The circle $S^1 := S_e^1$ is Γ -invariant and the circle S_g^1 corresponds to $g \times_{\Gamma} S^1$ in the representation $M_{\Gamma} = \mathbb{D}_8 \times_{\Gamma} S^1$.

Definition 2.15 *A weighted manifold $M = M_{\Gamma, u}$ is a formal disjoint sum*

$$\bigcup_{g \in G/\Gamma} u_g \otimes S_g^1$$

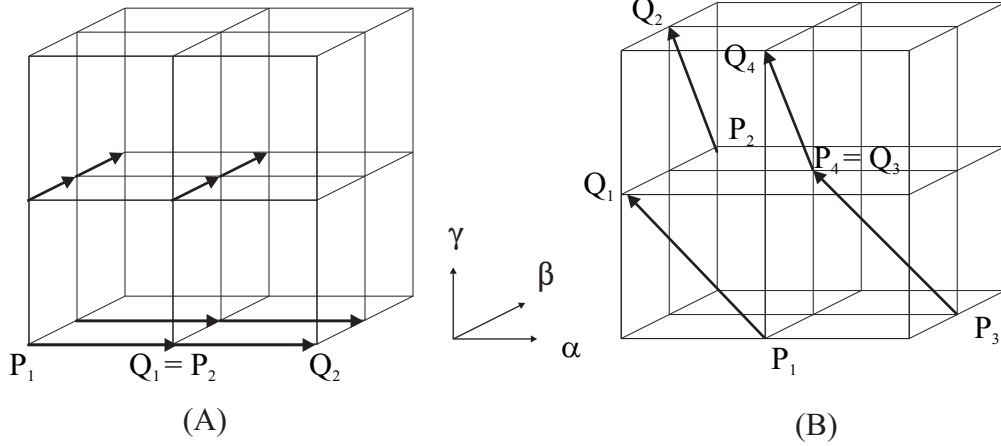


Figure 1: Geometric representatives of homology classes

where the union is taken over the representatives in the cosets and $u_g \in \mathcal{Z}$ for each $g \in G$. Moreover, if the function $u : G \rightarrow \mathcal{Z}$ is defined so that $u_g = u_h$ if g and h are in the same coset, then there is a compatibility condition $u_{gh} = g \cdot u_h = gh \cdot u_e$ for each $g, h \in G$. Usually $u_g \in \{t, -t\}$ where t is a preferred generator of the coefficient module \mathcal{Z} .

Example 2.16 Suppose that $\Gamma = \{1, \gamma\alpha, \alpha\beta, \gamma\beta\} \cong \mathbb{Z}/4$. Let $u_g := t$ for each $g \in \Gamma$ and $u_g := -t$ otherwise. Then the weights u_g obviously satisfy the compatibility condition and $M_{\Gamma, u} = t \otimes S^1 \cup (-t) \otimes S_\alpha^1$ is a weighted manifold.

Definition 2.17 A weighted manifold $M = M_{\Gamma, u}$ determines a well defined element $[M]$ in the group $H_1(\mathbb{D}_8, \mathcal{Z})$. If $u_g \in \{t, -t\}$ then M is called a special weighted manifold. In this case $[M]$ can be understood as a “fundamental class” of M which depends both on the orientation of M and the weight distribution $u : G \rightarrow \mathcal{Z}$. We say that $C \in H_1(\mathbb{D}_8, \mathcal{Z})$ is represented by a weighted manifold $M = M_{\Gamma, u}$ if $C = [M]$.

Proposition 2.18 The element $X_{ab} \in H_1(\mathbb{D}_8, \mathcal{Z})$ is represented by a special weighted manifold $M_{(\alpha\beta), u}$. The elements $\pm X_{ca}$ are both represented by special weighted manifolds of the form $M_{(\gamma\alpha), u} = u(e) \otimes S_e^1 \cup u(\alpha) \otimes S_\alpha^1$. If we choose a preferred generator $t \in \mathcal{Z}$ and the orientation on S_e^1 so that the action of $\gamma\alpha$ on S_e^1 is a rotation through the angle of 90° in the positive direction, then X_{ca} is represented by the weighted manifold with the weights $u_e = t$ and $u_\alpha = -t$ (Example 2.16). A representation of $-X_{ca}$ is obtained from the representation of X_{ca} by either changing the orientation of S_e^1 or by using the weights $u_e = -t$ and $u_\alpha = t$.

Proof: The proof is similar to the proof of Proposition 2.14. We start with the observation that one can associate to the generator X_{ca} the vector P_1Q_1 , in the usual model of the \mathbb{D}_8 -space $T^3 = (S^1)^3$, depicted in Figure 1 (B). Then this vector is completed to a circle $S_e^1 := P_1Q_1 + (\gamma\alpha)P_1Q_1 + (\alpha\beta)P_1Q_1 + (\gamma\beta)P_1Q_1$ which is assumed to be oriented in the direction of the vector P_1Q_1 . The proof of the corresponding statement for the element X_{ab} is similar so we omit the details. \square

Remark 2.19 Note that the special weighted manifold $M_{(\alpha\beta), u}$, representing the element X_{ab} , is a union of four circles. Since X_{ab} is an element of order 2, contrary to the case of the generator X_{ca} , one does not have to worry about either the orientation of S_e^1 or the precise value of $u_e \in \{t, -t\}$.

2.6.4 The algorithm

The weighted manifolds arise in applications as follows. Suppose that $f : N \rightarrow V$ is a \mathbb{D}_8 -equivariant map, where N is an oriented, free \mathbb{D}_8 -manifold and V a real, linear representation of the group \mathbb{D}_8 , such that

where $\epsilon_i := \text{sign}(\langle \alpha_i, X \rangle)$ and the letters a and b record the way each individual interval is partitioned. Obviously, in the case $(3m, 2m, 2)$, any sign vector and any ab -word of length $2m$ with the same number of occurrences of letters a and b , serves as the type of a unique solution (H_1, H_2) . Let us denote by \mathcal{T} the set of all types. The equivariant Poincaré dual to the obstruction class lies in the group $H_0((\mathbb{Z}/2)^{\oplus 2} \rtimes \mathbb{Z}/2, M) \cong \mathbb{Z}/2$ where M is one of the modules \mathbb{Z}, \mathcal{Z} described in the Section 2.3.3.

Observation: The dual of the obstruction class is a nonzero element in $H_0((\mathbb{Z}/2)^{\oplus 2} \rtimes \mathbb{Z}/2, M) \cong \mathbb{Z}/2$ if and only if the number of $G = (\mathbb{Z}/2)^{\oplus 2} \rtimes \mathbb{Z}/2$ orbits in the G -set \mathcal{T} is even.

Proposition 3.1 *The equivariant map $A : (S^{3m})^2 \rightarrow S((U_2)^{\oplus 2m})$ (Problem 2.3) exists if and only if $m = 2^q - 1$ for some integer q . It follows that a triple $(3 \cdot 2^q - 3, 2^{q+1} - 2, 2)$ is admissible, i.e. $\Delta(2^{q+1} - 2, 2) = 3 \cdot 2^q - 3$.*

Proof: The number of G -orbits in \mathcal{T} is $\frac{1}{2} \binom{2m}{m} = \binom{2m-1}{m-1}$. The result follows from the well known fact that $\binom{n}{k}_2 = 1$ iff the binary representation of k is a subword of the binary representation of n . \square

3.2 The case $\Delta = 1$

We have already observed in Section 3.1 that our method in the case $\Delta = 2d - 3j = 0$ does not provide too many interesting admissible triples. It turns out that the case $\Delta = 1$ is much more interesting from this point of view, so in this section we focus our attention on the triples of the form $(d, j, 2) = (3m + 2, 2m + 1, 2)$. Following essentially the notation from Section 2.2, we denote by $\mathcal{M} = \mathcal{M}[(\mu_\nu)_{\nu=1}^j]$ the space of all k -tuples (H_1, \dots, H_k) of oriented hyperplanes in \mathbb{R}^d which form an equipartition of the d -space with respect to each of the measures μ_ν . To be precise, we always work with the compactified configuration space $M_{\mathcal{P}} = (S^d)^k$, so one of the hyperplanes H_i is allowed to be the hyperplane “at infinity”, cf. Section 2.3.2.

In our case $(d, j) = (3m + 2, 2m + 1)$ and we assume that the set μ_1, \dots, μ_j of test measures is selected as in the Sections 2.4 and 3.2 as the measures concentrated and uniformly distributed on disjoint subintervals I_0, I_1, \dots, I_{2m} of the moment curve $\Gamma = \Gamma_{3m+2}$. Our goal is to describe the set \mathcal{M} of solutions to the equipartition problem in the case $(d, j) = (3m + 2, 2m + 1)$. Moreover, in order to be able to compute the relevant obstruction class $[\mathcal{M}]$ in one of the groups $H_1(\mathbb{D}_8; \mathbb{Z}) \cong \Omega_1(\mathbb{D}_8)$ or $H_1(\mathbb{D}_8; \mathcal{Z})$, we will need a more precise “inventory” of classes of these solutions.

3.2.1 Types and inventory of solutions

As in Section 3.1, it is convenient to record the *type* $\tau(H_1, H_2)$ of a solution pair (H_1, H_2) at least in some typical cases. As opposed to the case $\Delta = 0$, this time there is one more degree of freedom, arising from the fact that not all intersection points of hyperplanes H_1 and H_2 with the curve Γ are necessary for the equipartitions of given intervals. As a consequence \mathcal{M} turns out to be a 1-dimensional manifold. Figure 3 serves as a fairly good sample of “snapshots” from a “movie” which describes the solution manifold \mathcal{M} in the case $(d, j) = (5, 3)$. Each individual drawing in Figure 3 represents a single solution. More precisely, a drawing should be interpreted as an “orthogonal image” of the relevant part of moment curve Γ containing the intervals I_0, I_1, I_2 in the 2-plane P orthogonal to $H_1 \cap H_2$. We emphasize that the hyperplanes H_1 and H_2 , or rather their intersections with P , are represented on these pictures by the horizontal and vertical coordinate lines, while the associated orthogonal unit vectors are denoted by α and β .

The type $\tau(H_1, H_2)$ of the solution pair (H_1, H_2) , depicted in the first drawing, is by definition

$$\tau(H_1, H_2) = B(++)aab$$

which takes into account that

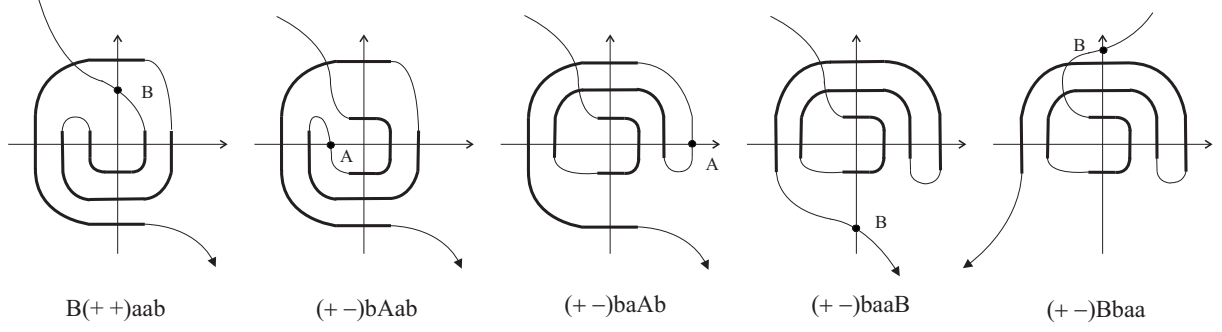


Figure 3: Metamorphoses of curves

- the three intervals I_0, I_1, I_2 are partitioned according to the word aab (cf. Section 3.1),
- the initial point of the interval I_0 belongs to the first quadrant and has $(++)$ for the associated sign vector,
- the *free* intersection point, i.e. the point (B on the picture) not used for partition of intervals, belongs to H_2 and precedes all intervals I_i .

The same bookkeeping scheme is used for describing the types of other solutions shown in Figure 3. For example $\tau(H_1, H_2) = (+-)bAab$ is the type of the solution depicted on second drawing in Figure 3.

Our next observation is that all solutions depicted in Figure 3 belong to the same connected component of the manifold \mathcal{M} . Indeed, these “metamorphoses” of solutions leading from one drawing to another, not visible in Figure 3, are shown in Figures 4 and 5. It turns out that there exist essentially two types of metamorphoses, the xy -type shown in Figure 4 (marked by the appearance of both letters a and b) and the opposite xx -type shown in Figure 5. The key observation is that the types of solutions obey a very simple law of transformation. As before, a *free* intersection point is a point $A \in \Gamma \cap H_1$ or $B \in \Gamma \cap H_2$ which does not belong to any of the intervals I_j .

Proposition 3.2 *The passage of a free point through an interval is recorded on Figures 4 and 5. The types of associated solutions change according to the following rules*

$$\begin{aligned}
 B(\epsilon_1, \epsilon_2)a &\longrightarrow (\epsilon_1, -\epsilon_2)bA \\
 A(\epsilon_1, \epsilon_2)b &\longrightarrow (-\epsilon_1, \epsilon_2)aB \\
 A(\epsilon_1, \epsilon_2)a &\longrightarrow (-\epsilon_1, \epsilon_2)aA \\
 B(\epsilon_1, \epsilon_2)b &\longrightarrow (\epsilon_1, -\epsilon_2)bB
 \end{aligned} \tag{27}$$

where A, B are free points, letters a, b record the type of a partition and $(\epsilon_1, \epsilon_2) \in \{+, -\}^2$ is a sign vector.

Proof: The proof is essentially by inspection of Figures 4 and 5. Note that the transformation rules are equivariant with respect to the group \mathbb{D}_8 which allows us to assume that $(\epsilon_1, \epsilon_2) = (+, +)$. \square

Remark 3.3 Proposition 3.2 records the change of the type of a solution if the free point passes over the first interval I_0 . This is the reason why we see the change of the sign vector (ϵ_1, ϵ_2) . Note however, that the same analysis yields similar rules for the change of types of solutions when a free point interchanges its position with some of the intervals I_i for $i \neq 1$

$$Ba \rightarrow bA \quad Ab \rightarrow aB \quad Aa \rightarrow aA \quad Bb \rightarrow bB \tag{28}$$

These changes do not involve the change of the associated sign vector. The reader should note that both (27) and (28) have the following simple interpretation. They both say that the underlying word does not change except for a shift of the capital letter one step to the right.

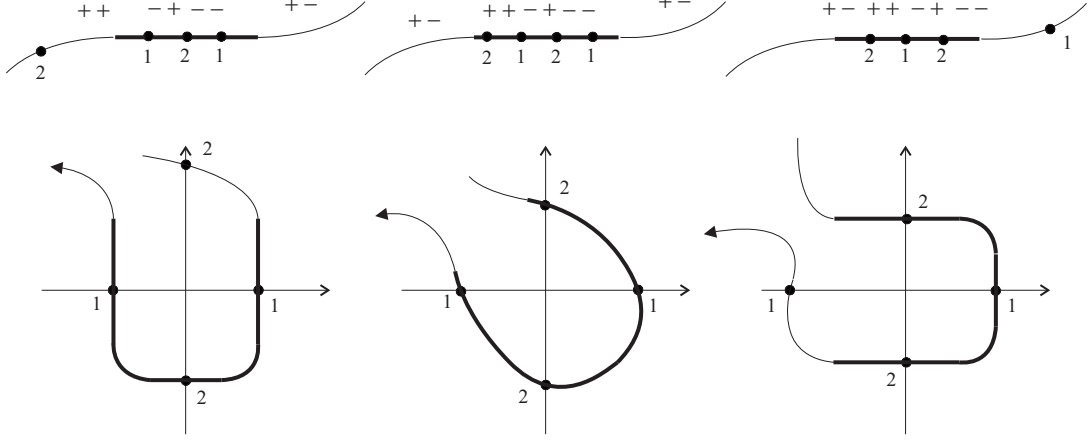


Figure 4: $B(++)a \rightsquigarrow (+-)bA$

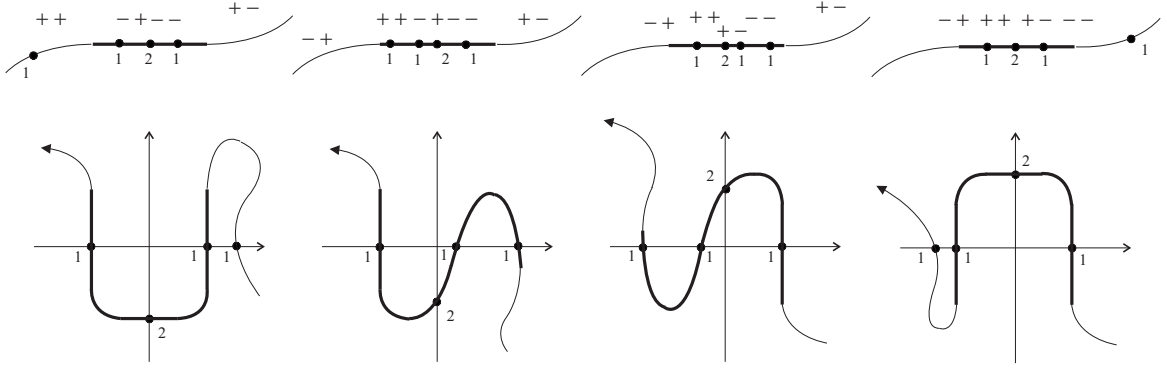


Figure 5: $A(++)a \rightsquigarrow (-+)aA$

The free point, denoted by a capital letter A or B in the type $\tau(H_1, H_2)$ of a solution (H_1, H_2) , moves to $+\infty$ on the moment curve Γ , passes through the infinite point and appears again on the other side and approaches intervals I_1, \dots, I_j from $-\infty$. This passage through an infinite point is formally justified by the fact that the point $(0, \dots, 0, 1) \in \mathbb{R}^{d+1}$ determines the point $+\infty = -\infty$ which can be seen as an infinite point on the moment curve Γ compactified configuration space $M_{\mathcal{P}} = (S^d)^k$

Example 3.4 Here is a complete sequence of types of solutions in the connected component of the solution manifold $\mathcal{M} = \mathcal{M}_{(5,3)}$ which contains all solutions depicted in the Figure 3.

$$\begin{array}{llll}
 \tau_1 = B(++)aab & \tau_5 = B(+-)baa & \tau_9 = A(++)bba & \tau_{13} = A(-+)abb \\
 \tau_2 = (+-)bAab & \tau_6 = (++)bBaa & \tau_{10} = (-+)aBba & \tau_{14} = (++)aAbb \\
 \tau_3 = (+-)baAb & \tau_7 = (++)bbAa & \tau_{11} = (-+)abBa & \tau_{15} = (++)aaBb \\
 \tau_4 = (+-)baaB & \tau_8 = (++)bbaA & \tau_{12} = (-+)abbA & \tau_{16} = (++)aabB
 \end{array} \tag{29}$$

Note that $\tau_{17} = \tau_1 = B(++)aab$ while the types $\tau_1, \tau_2, \dots, \tau_5$ correspond to the solutions depicted in Figure 3. The list (29) can be conveniently abbreviated as follows

$$BAAB(+-) \quad BBAA(++) \quad ABBA(-+) \quad AABB(++) \tag{30}$$

Definition 3.5 A signed word is a word of the form $w(\epsilon_1, \epsilon_2)$, where $w = x_1x_2 \dots x_{2r}$ is a balanced $\{A, B\}$ -words in the sense of Section 2.5. Two signed words, $w_1(\epsilon_1, \epsilon_2)$ and $w_2(\eta_1, \eta_2)$ are cyclically equivalent if one can be obtained from the other by subsequent applications of the following rules

$$Aw(\epsilon_1, \epsilon_2) \longleftrightarrow wA(-\epsilon_1, \epsilon_2) \quad Bw(\epsilon_1, \epsilon_2) \longleftrightarrow wB(\epsilon_1, -\epsilon_2) \quad (31)$$

Cyclic equivalence is an equivalence relation, and an equivalence class is called a cyclic signed word. The cyclic signed word, associated to a signed word $C = w(\epsilon_1, \epsilon_2)$ is denoted by $[C] = [w(\epsilon_1, \epsilon_2)]$.

Note that the cyclic signed word represents a combinatorial counterpart of a connected component in a solution manifold. More precisely we have the following observation.

Observation 3.6 Example 3.4 clearly shows that each circle, representing a connected component of the solution manifold $\mathcal{M} = \mathcal{M}_{d,j}$, can be encoded by an appropriate *cyclic sign word*. Conversely, each cyclic signed word is associated to some circle in the corresponding solution manifold. The one to one correspondence between circles in solution manifolds and cyclic signed words, is \mathbb{D}_8 -equivariant in the following sense. The group \mathbb{D}_8 , which acts on the solution manifold \mathcal{M} , acts also both on signed words and on their cyclic equivalence classes. The action on signed words is described as follows

$$\alpha(w(\epsilon_1, \epsilon_2)) = w(-\epsilon_1, \epsilon_2) \quad \beta(w(\epsilon_1, \epsilon_2)) = w(\epsilon_1, -\epsilon_2) \quad \gamma(w(\epsilon_1, \epsilon_2)) = w^*(\epsilon_2, \epsilon_1) \quad (32)$$

where the conjugation $w \mapsto w^*$ is the operation described in Definition 2.5.

Definition 3.7 The \mathbb{D}_8 -orbit of a cyclic signed word is called a generating class of words or simply a generating class. The one to one correspondence between cyclic signed words and circles in solution manifolds, extends to the correspondence between generating classes and minimal \mathbb{D}_8 -invariant submanifolds of the solution manifold. These minimal \mathbb{D}_8 -invariant submanifolds have, according to Propositions 2.14 and 2.18, the form M_Γ , for an appropriate subgroup $\Gamma \subset \mathbb{D}_8$, and they are potential carriers of nontrivial obstruction classes X, Y, Z , respectively $X_{ab}, \pm X_{ca}$.

Example 3.8 The \mathbb{D}_8 -orbit of the cyclic signed word (30), i.e. the corresponding generating class of words, is exhibited in the following table.

$$\begin{array}{cccc} AAB B + + & AAB B + - & AAB B - + & AAB B - - \\ AB B A - + & AB B A - - & AB B A + + & AB B A + - \\ BB A A + + & BB A A + - & BB A A - + & BB A A - - \\ BA A B + - & BA A B + + & BA A B - - & BA A B - + \end{array} \quad (33)$$

The cyclic signed word (30) is exhibited as the first column in the table (33). It is easy to check that its stabilizer is the group $(\gamma) \subset \mathbb{D}_8$. We infer from here that the corresponding submanifold of the solution manifold $\mathcal{M}_{(5,3)}$ is of the type $M_{(\gamma)}$, in the notation of Proposition 2.14. Hence, the generating class (33) contributes the element Y to the associated homology or bordism obstruction class $D(\omega)$ described in Section 2.6. It is easily checked that in the case $(d, j) = (5, 3)$ there is exactly one more generating class of words shown on the following table,

$$\begin{array}{cc} ABAB + + & ABAB + - \\ BABA - + & BABA - - \\ ABAB - - & ABAB - + \\ BABA + - & BABA + + \end{array} \quad (34)$$

The stabilizer of the first column is $(\alpha\beta)$ which, according to Proposition 2.14, represents a trivial element in the group $\Omega_1(\mathbb{D}_8)$.

The final conclusion is that the obstruction $[\mathcal{M}] = [\mathcal{M}_{(5,3)}] = Y$ is nontrivial which implies that the triple $(5, 3, 2)$ is admissible.

3.2.2 The $(8, 5, 2)$ case

Theorem 3.9 *The triple $(8, 5, 2)$ is admissible. In other words for each collection of 5 continuous mass distributions μ_1, \dots, μ_5 in \mathbb{R}^8 there exist two hyperplanes H_1 and H_2 forming an equipartition for each of the measures μ_i .*

Proof: The first obstruction X for the existence of a \mathbb{D}_8 -equivariant map $f : S^8 \times S^8 \rightarrow U_2^{\oplus 5}$ is an element of the group $H_1(\mathbb{D}_8, \mathcal{Z})$. It turns out that in the $(8, 5, 2)$ -case there exist precisely three generating classes (Definition 3.7) of signed words, displayed on the diagrams (35), (36) and (37) respectively.

$$\begin{array}{ll}
 AAABBB++ & AAABBB+- \\
 AABBBB-+ & AABBBB-- \\
 ABBBBAA++ & ABBBBAA+- \\
 BBBBAAA-+ & BBBBAAA-- \\
 BBAAAAB-- & BBAAAAB-+ \\
 BAAABBB-+ & BAAABBB-- \\
 AAABBBB-- & AAABBBB-+ \\
 AABBBBA+- & AABBBBA++ \\
 ABBBBAA-- & ABBBBAA-+ \\
 BBBBAAA+- & BBBBAAA++ \\
 BBAAAAB++ & BBAAAAB+- \\
 BAAABBB+- & BAAABBB++
 \end{array} \tag{35}$$

$$\begin{array}{cccc}
 AABABB++ & AABABB+- & AABBAB++ & AABBAB+- \\
 \cdot & \cdot & \cdot & \cdot \\
 BABBAA++ & BABBAA+- & BBABAA++ & BBABAA+- \\
 \cdot & \cdot & \cdot & \cdot \\
 BBAABA-- & BBAABA-+ & ABAABB++ & ABAABB+- \\
 \cdot & \cdot & \cdot & \cdot \\
 AABABB-- & AABABB-+ & AABBAB-- & AABBAB-+ \\
 \cdot & \cdot & \cdot & \cdot \\
 BABBAA-- & BABBAA-+ & BBABAA-- & BBABAA-+ \\
 \cdot & \cdot & \cdot & \cdot \\
 BBAABA++ & BBAABA+- & ABAABB-- & ABAABB-+ \\
 \cdot & \cdot & \cdot & \cdot
 \end{array} \tag{36}$$

$$\begin{array}{ll}
 ABABAB++ & ABABAB+- \\
 BABABA-+ & BABABA-- \\
 ABABAB-- & ABABAB-+ \\
 BABABA+- & BABABA++
 \end{array} \tag{37}$$

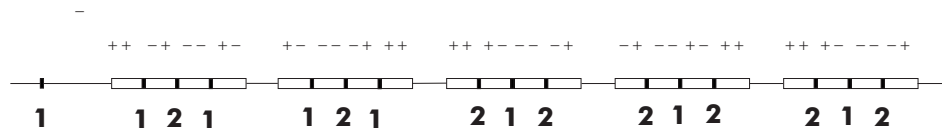
If M_1, M_2 and M_3 are the associated minimal \mathbb{D}_8 -manifolds and $[M_i]$ are the associated weighted fundamental classes, then $X = [M_1] + [M_2] + [M_3]$. Since M_2 has 4 connected components we conclude that $[M_2] = X_{ab} = 2X_{ca}$. We will show now that $[M_1]$ and $[M_3]$ cancel out following the procedure described by the algorithm in Section 2.6.4.

Let C_1 be the circle corresponding to the first column in the diagram (35), oriented following the lexicographical order of signed words. Note that the element $\gamma\alpha \in \mathbb{D}_8$ rotates this circle through the angle of 270° . Similarly, C_2 is the circle corresponding to the first column in the diagram (37) oriented by the same rule. Note that this circle is also rotated by $\gamma\alpha$ through the angle of 270° .

Following the procedure described in Section 2.6.4, we should choose orientations T and t on $N = S^8 \times S^8$ and on $U_2^{\oplus 5}$. Let us suppose that aside from the intervals $I_i = [a_i, b_i]$ on the moment curve Γ_8 , representing the measures μ_i , we choose one more (open) interval $J_0 = (a_0, b_0)$ such that $b_0 < a_1$. Here for simplicity we identify Γ_8 with the associated parameter space \mathbb{R} . Let $J_1 := (a_1, b_5)$ and choose a point z in the interval (b_0, a_1) . Define $\mathcal{A}_1 \subset S^8$ as the space of all oriented hyperplanes H_1 such that z is in the positive halfspace H_1^+ and $H_1 \cap \Gamma_8$ consists of precisely 8 distinct points $x_1 < x_2 < \dots < x_8$ such that $x_1 \in J_0$ and $x_i \in J_1$ for each $i \geq 2$. Similarly, let \mathcal{A}_2 be a collection of all hyperplanes H_2 such that again z is in the positive halfspace H_2^+ and the intersection $H_2 \cap \Gamma_8$ consists of 8 distinct points $y_1 < y_2 < \dots < y_8$ but this time $y_i \in J_1$ for each $i \geq 1$. Both \mathcal{A}_1 and \mathcal{A}_2 are open (convex) cells in S^8 .

$b_1^+ \ b_1^- \ b_1^+ \ b_2^+ \ b_2^- \ b_2^+ \ b_3^+ \ b_3^- \ b_3^+ \ b_4^+ \ b_4^- \ b_4^+ \ b_5^+ \ b_5^- \ b_5^+$

x_2	1	(-1)				
x_3	-1		1			
x_4						
x_5		-1	(1)			
x_6			1			
x_7				-1		
x_8					1	
y_1	1					
y_2		-1				
y_3			1	(-1)		
y_4				-1		
y_5					1	
y_6				-1	(1)	
y_7					1	(-1)
y_8						-1



Note that by fixing the coordinate x_1 , the signed words $AAABBB++$ and $ABABAB++$ can be interpreted as points on circles C_1 and C_2 respectively. In both cases $x_2, \dots, x_8, y_1, \dots, y_8$ can be chosen as the coordinates in the normal slices to circles C_1 and C_2 at these points. Following the procedure from Section 2.6.4 we should compare the orientations in the normal bundles to these points with the orientation t . This amounts to computing the signs of the corresponding Jacobian matrices, Figures 6 and 7. The circled entries in these matrices can be removed by elementary column/row operations, so they have the same determinant as the corresponding signed, permutation matrices.

3.2.3 The case $(6m + 2, 4m + 1, 2)$

The following definition introduces the key combinatorial functions which permit us to translate the analysis given in Section 2.6.4 into an algorithm for computing the relevant obstruction class.

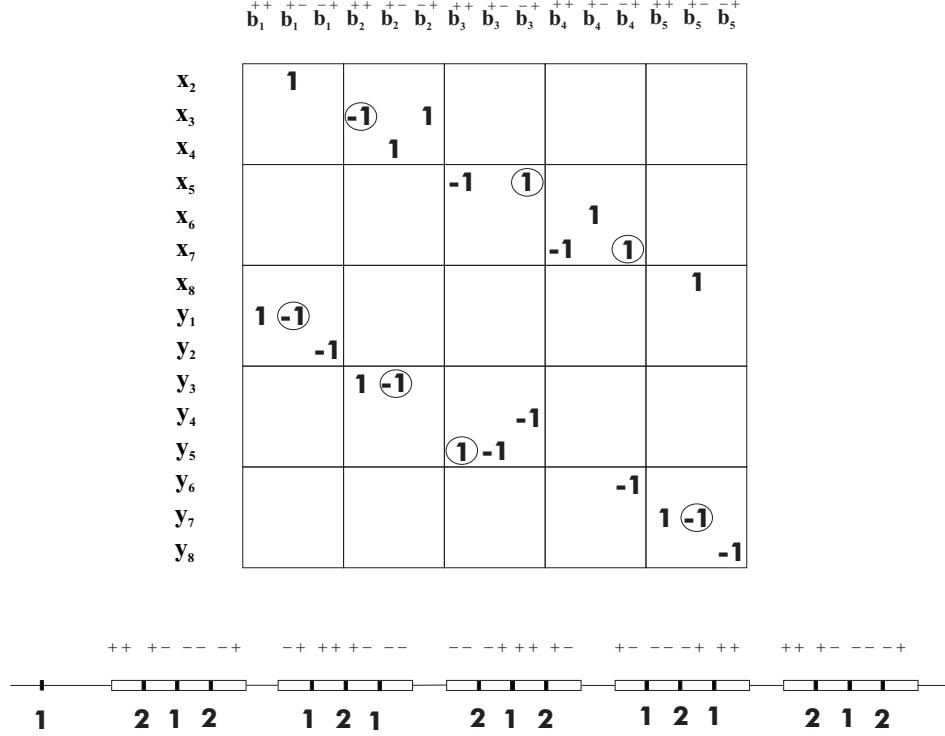


Figure 7: Jacobian matrix: $ABABAB$ -case

Definition 3.10 Let $\beta(n)$ be the number of generating classes of cyclic signed words of length $2n$ which consist of 4 different cyclic signed words. Suppose that G is a generating class of cyclic signed words consisting of precisely 2 cyclic signed words and let $\omega = C_1 C_2 \dots C_{2n} ++$ be the lexicographically first signed $\{A, B\}$ -word in this class. Assume that this cyclic signed word is oriented according to the action of the cyclic group $\mathbb{Z}/(2n)$ permuting its letters. Let $\epsilon(G)$ be $+1$ or -1 depending on whether the action of the generator $\gamma\alpha$, a “rotation” through 90° , agrees with this orientation or not. Associate a Jacobian matrix $J(\omega)$ to this word by a direct generalization of the algorithm for constructing these matrices, described in the proof of Theorem 3.9 where it led to Figures 6 and 7. Let $\eta(G)$ be the sign of the determinant of this matrix. Define $\alpha(n)$ (respectively $\beta(n)$) as the number of generating classes G with two signed cyclic word components such that $\epsilon(G) = \eta(G)$ (respectively $\epsilon(G) \neq \eta(G)$).

Remark 3.11 Note that $\alpha(n), \beta(n), \gamma(n)$ are combinatorial functions which have much in common with very well known functions enumerating the number of cyclic words in a given alphabet. However, an explicit formula for these functions, or at least for $\alpha(n)$ and $\gamma(n)$ remains an interesting open problem.

Proof and comments on Theorem 1.4: The proof of Theorem 1.4 follows step by step the procedure outlined in the proof of the special case $(8, 5, 2)$. The computation of the associated Jacobian matrices can be simplified as follows. We illustrate the idea on the special case of the matrix associated to the $AAABBB$ -case of the solution manifold associated to the triple $(8, 5, 2)$, Figure 6. Instead of working with $x_2, \dots, x_8, y_1, \dots, y_8$, as local coordinates, we could put these functions in the order of appearance, relative the equipartition of intervals shown in Figure 6. By inspection of Figure 6 we observe that the natural order is

$$x_2 \ y_1 \ x_3 \ x_4 \ y_2 \ x_5 \ y_3 \ x_6 \ y_4 \ y_5 \ x_7 \ y_6 \ y_7 \ x_8 \ y_8.$$

This system of functions has an advantage that the Jacobian matrix with respect to this system is a 3×3 -block diagonal matrix. Note that the sign of this matrix is equal to the sign of the matrix on Figure 6, multiplied by the sign of the corresponding shuffle permutation, associated to the word $AAABBB$. \square

3.2.4 The case $(6m - 1, 4m - 1, 2)$

For completeness we discuss here the case of a triples $(6m - 1, 4m - 1, 2)$ although these results have an alternative proof based on completely different ideas, Section 4.

Proposition 3.12 *Suppose that $(d, j, k) = (6m - 1, 4m - 1, 2)$ where m is a positive integer. Then there exists an equipartition of $j = 4m - 1$ mass distributions in $R^d = R^{6m-1}$ if an element $o = o_1 + o_2 \in \mathbb{Z}/2 \oplus \mathbb{Z}/2$ is nonzero, where o_1 and o_2 are determined by the following congruences modulo 2,*

$$o_1 \equiv_2 O_1(m) := \sum_{\substack{k|2m \\ k \text{ is odd}}} A(k) \quad o_2 \equiv_2 O_2(m) := \sum_{\substack{k|2m \\ k \text{ is even}}} A(k) \quad (38)$$

where $A(k)$ is the number of $*$ -primitive, circular words of length $2k$, Definition 2.8.

Proof: The proof is a generalization of the analysis given in Example 3.8. In the case $(d, j, k) = (6m - 1, 4m - 1, 2)$ the relevant obstruction lives in the group $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. In order to compute this obstruction we “list” all generating classes of cyclic signed words of length $4m$ and determine the corresponding stabilizers, which allows us to apply Proposition 2.14. Observe that neither (α) nor (β) appear as stabilizers of connected components of generating classes of cyclic signed words. On the other hand the groups (γ) , $(\alpha\beta\gamma)$, respectively $(\alpha\gamma)$, $(\beta\gamma)$ do appear as stabilizers in the generating classes corresponding to $*$ -primitive, circular words and it is not difficult to distinguish these two cases. Suppose that $[w]$ is the generating class of a special word $w = (aa^*) \dots (aa^*)$ where aa^* is a $*$ -primitive word of length $2k$. Then a component of $[w]$ is stabilized by (γ) , respectively by $(\alpha\gamma)$, depending on whether k is even or odd. In light of Proposition 2.14 this immediately leads to formula (38). \square .

Corollary 3.13

$$\Delta(2^{q+1} - 1, 2) = 3 \cdot 2^q - 1.$$

Proof: It follows from Propositions 3.12 and 2.11 that only in the case $m = 2^p$ the obstruction element $o = o_1 + o_2 \in \mathbb{Z}/2 \oplus \mathbb{Z}/2$ is nonzero. The details are left to the reader. \square

4 Cohomological Methods

4.1 Ideal valued cohomological index theory

A standard tool for proving non existence of equivariant maps is a cohomological index theory, [14] [16] [21] [47]. A particularly useful form of this theory is the so called ideal valued cohomological index theory developed by E.Fadell and S. Husseini [17], see also [22] and [48].

In this section we demonstrate how this theory can be applied to the equipartition problem and compare it with the obstruction theory approach from previous sections.

Theorem 4.1 *Let*

$$\mathbb{P}_k = \text{Det} \begin{bmatrix} x_1 & x_1^2 & x_1^4 & \dots & x_1^{2^{k-1}} \\ x_2 & x_2^2 & x_2^4 & \dots & x_2^{2^{k-1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_k & x_k^2 & x_k^4 & \dots & x_k^{2^{k-1}} \end{bmatrix} \in \mathbb{F}_2[x_1, \dots, x_k] \quad (39)$$

be a Dickson polynomial. Then (d, j, k) is an admissible triple if

$$(\mathbb{P}_k)^j \notin \text{Ideal}\{x_1^{d+1}, \dots, x_k^{d+1}\}. \quad (40)$$

Proof: The proof is based on the ideal valued, cohomological index theory, as developed by E. Fadell and S. Husseini, [17]. Recall that the equipartition problem can be reduced, Section 2.3.2, to the question of the (non)existence of a W_k -equivariant map $A : (S^d)^k \rightarrow S(U_k^{\oplus j})$ where $W_k = (\mathbb{Z}/2)^{\oplus k} \rtimes S_k$, U_k is a W_k -representation described in Section 2.3.2 and $S(V)$ is a unit sphere in V . The representation U_k was characterized by the property that its restriction on the subgroup $H = (\mathbb{Z}/2)^{\oplus k}$ is equivalent to the regular representation of the group $(\mathbb{Z}/2)^{\oplus k}$.

The cohomology of the classifying space BH with \mathbb{F}_2 -coefficients is $H^*(BH; \mathbb{F}_2) = \mathbb{F}_2[x_1, \dots, x_k]$ and the corresponding indices are

$$\text{Index}^H \left((S^d)^k \right) = (x_1^{d+1}, \dots, x_k^{d+1}),$$

$$\text{Index}^H \left(S \left(U_k^{\oplus j} \right) \right) = \left((\mathbb{P}_k(x_1, \dots, x_k))^j \right),$$

where

$$\mathbb{P}_k(x_1, \dots, x_k) = x_1 \cdots x_k (x_1 + x_2) \cdots (x_{k-1} + x_k) \cdots (x_1 + \cdots + x_k)$$

is a Dickson polynomial. It is well known [42] that \mathbb{P}_k can be expressed in the form of determinant (39), or more explicitly

$$\mathbb{P}_k(x_1, \dots, x_k) = \sum_{\sigma \in \Sigma_k} x_{\sigma(1)}^{2^{k-1}} x_{\sigma(2)}^{2^{k-2}} \cdots x_{\sigma(k)}.$$

A sufficient condition, [17], for a nonexistence of a H -equivariant map $f : X \rightarrow Y$ is the relation $\text{Index}^H(Y) \not\subseteq \text{Index}^H(X)$. In our case this relation takes the form of the condition (40) and the result follows. \square

In order to apply Theorem 4.1 we search in $(\mathbb{P}_k(x_1, \dots, x_k))^j$ for a summand where the biggest exponent is as small as possible. From the properties of the binomial coefficients over \mathbb{F}_2 we deduce that the summands we are looking for are of the form

$$m = \left(x_1^{2^{k-1}} x_2^{2^{k-2}} \cdots x_k \right)^{2^q} \cdot \left(x_1 x_2^2 \cdots x_k^{2^{k-1}} \right)^r,$$

where $j = 2^q + r$ and $0 \leq r \leq 2^q - 1$. It is not difficult to see that there exists a summand with the exponent $2^{k+q-1} + r$. If $2^{k+q-1} + r \leq d$ then the condition (40) from Theorem 4.1 is fulfilled and as a consequence we obtain the following result.

Theorem 4.2

$$\Delta(2^q + r, k) \leq 2^{k+q-1} + r. \quad (41)$$

It is interesting to compare the inequality (41) with the only existing general upper bound $\Delta(j, k) \leq j2^{k-1}$ from [33]. Since

$$(2^q + r)2^{k-1} - 2^{k+q-1} - r = r(2^{k-1} - 1) \geq 0,$$

our estimate is the same as the one from [33] in the case $r = 0$ and strictly better in all other cases. In the special case $k = 2$ we obtain the inequalities

$$3 \cdot 2^{q-1} + \frac{3r}{2} \leq \Delta(2^q + r, 2) \leq 2^{q+1} + r.$$

Note that the best estimate is obtained when j is slightly less than a power of 2. For example if $r = 2^q - 1$, i.e. if $j = 2^{q+1} - 1$, we obtain the exact value

$$\Delta(2^{q+1} - 1, 2) = 3 \cdot 2^q - 1,$$

which is a result already obtained in Section 3.2.4 (Corollary 3.13). Also, we obtain almost precise values for $\Delta(j, 2)$ when $j = 2^{q+1} - 2$ or $j = 2^{q+1} - 3$

$$3 \cdot 2^q - 3 \leq \Delta(2^{q+1} - 2, 2) \leq 3 \cdot 2^q - 2 \quad 3 \cdot 2^q - 4 \leq \Delta(2^{q+1} - 3, 2) \leq 3 \cdot 2^q - 3. \quad (42)$$

Note that we already know that $\Delta(2^{q+1} - 2, 2) = 3 \cdot 2^q - 3$ (Proposition 3.1). Also note that the result $\Delta(5, 2) = 8$ (Theorem 3.9) cannot be obtained by this method since the general upper bound (42) implies only the inequality $\Delta(5, 2) \leq 9$, already known to Ramos [33].

Here are some particular examples which illustrate the power of Theorem 4.2. The best previously known upper bounds are given in parentheses.

$$\begin{array}{ll} 17 \leq \Delta(7, 3) \leq 19 \quad (28) & 14 \leq \Delta(6, 3) \leq 18 \quad (24) \\ 35 \leq \Delta(15, 3) \leq 39 \quad (60) & 33 \leq \Delta(14, 3) \leq 38 \quad (56) \\ 27 \leq \Delta(7, 4) \leq 35 \quad (56) & 23 \leq \Delta(6, 4) \leq 34 \quad (48) \\ 57 \leq \Delta(15, 4) \leq 71 \quad (120) & 53 \leq \Delta(14, 4) \leq 70 \quad (112) \end{array} \quad (43)$$

References

- [1] M. Aigner. *Combinatorial Theory*, Springer-Verlag, Berlin 1979.
- [2] C.A. Athanasiadis, J. Rambau, and F. Santos. The generalized Baues problem for cyclic polytopes II, *Publ. de l'Institut Mathématique*, 66(80) (1999), 3–15.
- [3] N. Alon, Splitting necklaces, *Advances in Math.* 63: 247–253, 1987.
- [4] N. Alon, Some recent combinatorial applications of Borsuk-type theorems, Algebraic, Extremal and Metric Combinatorics, M.M. Deza, P. Frankl, D.G. Rosenberg, editors, *Cambridge Univ. Press*, Cambridge 1988, pp. 1–12.
- [5] I. Bárány, Geometric and combinatorial applications of Borsuk's theorem, New trends in Discrete and Computational Geometry, János Pach, ed., *Algorithms and Combinatorics 10*, Springer-Verlag, Berlin, 1993.
- [6] I. Bárány, J. Matoušek. Simultaneous partitions of measures by k -fans, *Discrete Comput. Geom.* (to appear).
- [7] P. Billingsley. *Convergence of Probability Measures*, John Wiley & Sons, 1968.
- [8] A. Björner, Topological methods, In R. Graham, M. Grötschel, and L. Lovász, editors, *Handbook of Combinatorics*. North-Holland, Amsterdam, 1995.
- [9] K. Borsuk. Drei Sätze über die n -dimensionale euklidische Sphäre. *Fundamenta Mathematicae*, 20:177–190, 1933.
- [10] G.E. Bredon. *Topology and Geometry*. Graduate Texts in Mathematics 139, Springer (1995).
- [11] K.S. Brown. *Cohomology of Groups*, Springer 1982.
- [12] P.E. Conner and E.E. Floyd. *Differentiable Periodic Maps*, *Ergebnisse der Mathematik und Ihrer Grenzgebiete Neue Folge* 33, Springer 1964.
- [13] T. tom Dieck, Transformation groups, *de Gruyter stud. in math.* 8, Berlin, 1987.
- [14] A. Dold. Parametrized Borsuk-Ulam theorems, *Comment. Math. Helv.*, vol. 63, 1988, pp. 275–285.
- [15] J. Eckhoff. Helly, Radon and Carathéodory type theorems, *Handbook of Convex Geometry*, P.M. Gruber and J.M. Wills (eds.), North-Holland, Amsterdam, vol. A, 1993, pp. 389–448.
- [16] E. Fadell and S. Husseini. Index theory for G -bundle pairs with applications to Borsuk-Ulam type theorems for G -sphere bundles, in *Nonlinear Analysis*, pp. 307–337, World Scientific, Singapore (1987).
- [17] E. Fadell and S. Husseini. An ideal-valued cohomological index theory with applications to Borsuk-Ulam and Bourgin-Yang theorems, *Ergod. Th. and Dynam. Sys.*, vol. 8*, 1988, pp. 73–85.
- [18] I.M. Gessel and R.P. Stanley. Algebraic Enumeration, In R. Graham, M. Grötschel, and L. Lovász, editors, *Handbook of Combinatorics*. North-Holland, Amsterdam, 1995.
- [19] B. Grünbaum. Partitions of mass-distributions and convex bodies by hyperplanes, *Pacific J. Math.*, 10 (1960), 1257–1261.
- [20] H. Hadwiger. Simultane Verteilung zweier Körper, *Arch. Math.* (Basel), 17 (1966), 274–278.
- [21] M. Izydorek, S. Rybicki. On parametrized Borsuk-Ulam theorem for free Z_p -action, *Proc. Barcelona Conf. on Alg. Topology 1990*, pp. 227–234, Lecture Notes in Mathematics 1509, Springer (1992).
- [22] J. Jaworowski. A continuous version of the Borsuk-Ulam theorem. *Proc. Amer. Math. Soc.*, vol. 82 (1981), pp. 112–114.
- [23] V. Krishnamurthy. *Combinatorics, Theory and Applications*, East-West Press 1985.
- [24] L. Lovász. Kneser's conjecture, chromatic number and homotopy, *J. Combin. Theory A*, 25, 319–324.
- [25] V.V. Makeev. On some combinatorial geometric problems on vector bundles (in Russian). *Algebra and analysis*, 14 (2002), pp. 169–191.
- [26] J. Matoušek. Efficient partition trees, *Discrete Comput. Geom.* 8 (1992), 315–334.
- [27] J. Matoušek. Range searching with efficient hierarchical cuttings, *Discrete Comput. Geom.* 10 (1993), 157–182.
- [28] J. Matoušek. Geometric Range Searching, *Report B 93-09*, Institute for Computer Science, Department of Mathematics, Freie Universität, Berlin, July 1993.
- [29] J. Matoušek. *Using the Borsuk-Ulam Theorem*. Lectures on Topological Methods in Combinatorics and Geometry. Springer 2003.
- [30] J. Milnor and J.D. Stasheff. *Characteristic Classes*, Princeton University Press, Princeton (1974).
- [31] J. Pach (Ed.). *New Trends in Discrete and Computational Geometry*, Algorithms and Combinatorics 10, Springer (1993).
- [32] R. Rado. A theorem on general measure, *J. London Math. Soc.*, vol. 26 (1946), pp. 291–300.
- [33] E.A. Ramos. Equipartitions of mass distributions by hyperplanes, *Discrete Comput. Geom.*, vol. 15 (1996), pp. 147–167.

- [34] R.P. Stanley. *Enumerative Combinatorics*. Cambridge University Press, Vol. 1 1997, Vol. 2 1999.
- [35] H. Tverberg, S. Vrećica. On generalizations of Radon's theorem and the ham sandwich theorem, *Europ. J. Combinatorics*, vol. 14 (1993), pp. 259–264.
- [36] S. Vrećica. Tverberg's conjecture. *Discrete Comput. Geom.*
- [37] S. T. Vrećica and R. T. Živaljević. The ham-sandwich theorem revisited, *Israel J. Math.*, **78** (1992), 21–32.
- [38] S. T. Vrećica and R. T. Živaljević. Conical equipartitions of mass distributions. *Discrete Comput. Geom.* 25:335–350, 2001.
- [39] S. Vrećica, R. Živaljević. New cases of the colored Tverberg theorem, Jerusalem Combinatorics '93, H. Barcelo, G. Kalai (eds.) *Contemporary mathematics*, A.M.S. Providence 1994.
- [40] C.T.C. Wall. *Surgery on Compact Manifolds* (Second Edition), Mathematical Surveys and Monographs Vol. 69, A.M.S. 1999.
- [41] D.E. Willard. Polygon retrieval, *SIAM J. Comput.*, 11 (1982), 149–165.
- [42] C. Wilkerson. A primer on the Dickson invariants. *Proceedings of the Northwestern Homotopy theory conference, Contemp. Math.* 19 (1983), 421–434.
- [43] F. Yao, D. Dobkin, H. Edelsbrunner, M. Paterson. Partitioning space for range queries, *SIAM J. Comput.*, 18 (1989), 371–384.
- [44] A.C. Yao, F.F. Yao. A general approach to d -dimensional geometric queries, in *Proceedings of the 17th ACM Annual Symposium on the Theory of Computing*, 1985, 163–169.
- [45] *Lectures on Polytopes*, Graduate Texts in Mathematics, Springer 1995.
- [46] R. Živaljević. Topological methods, in *CRC handbook of discrete and computational geometry*, J.E. Goodman, J. O'Rourke (eds.), CRC Press, New York (1997).
- [47] R. Živaljević. User's guide to equivariant methods in combinatorics, *Publ. Inst. Math. Belgrade*, vol. 59(73) (1996), pp. 114–130.
- [48] R. Živaljević. User's guide to equivariant methods in combinatorics II, *Publ. Inst. Math. Belgrade*, vol. 64(78) (1998), pp. 107–132.
- [49] R.T. Živaljević, The Tverberg–Vrećica problem and the combinatorial geometry on vector bundles, *Israel J. Math.*, 111:53–76, 1999.
- [50] R. Živaljević, Combinatorics of topological posets: Homotopy complementation formulas, *Adv. Appl. Math.* 21 (1998), 547–574.
- [51] R. Živaljević, S. Vrećica. An extension of the ham sandwich theorem, *Bull. London Math. Soc.*, vol. 22 (1990) pp. 183–186.
- [52] R. Živaljević, S. Vrećica. The colored Tverberg's problem and complexes of injective functions, *J. Combin. Theory, Ser. A*, vol. 61 (2) (1992), pp. 309–318.